

in the stress σ_{s_z} . The applied theory introduces a discrepancy of the same order as the quantity being considered in calculating the stress τ_{s_z} in the slab bending case.

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REPRESENTATION OF SOLUTIONS OF THE GREEN TYPE FOR EQUATIONS OF SHELLS BY THE SMALL PARAMETER METHOD

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A method is presented for asymptotic integration of equations in the theory of shells (convex shells are examined) for the case where the free terms in the equations consist of a Dirac delta function or its derivatives. These solutions, which are solutions represented by a function of the Green type, correspond to the action of concentrated forces or moments on the shell.

At first the analysis is carried out for one equation and then it is shown how the obtained results are extended to the system.

1. Let us examine the linear differential equation containing the small parameter ε which appears in the theory of shells as the relative thickness

$$\varepsilon^s M(w) + L(w) = \delta \quad (1.1)$$

Here M and L are elliptic differential operators with variable coefficients and highest derivatives of orders $2m$ and $2l$, $m > l$, respectively. Without any loss we can write $s = 2(m - l)$.

In the theory of shells the order of operators M and L are equal to $2m = 8$ and $2l = 4$; however, all arguments will be carried out for arbitrary m and l . Since the dimension of the space n does not have any significance with respect to the presented arguments, we shall carry them out for any arbitrary even n (the case of uneven n is examined in an analogous manner).

For this equation we shall construct a solution making use of the method of asymptotic integration [1-3]. Such solutions are called fundamental, and the basic content of this work consists in the adaptation of the rigorous process of the asymptotic method to the construction of these solutions.

The method of asymptotic integration consists of the following: the solution is represented in the form of a sum of two solutions of the so-called slowly varying fundamental type and solutions of the edge effect type [1] (in papers [2 and 3] the latter are referred to as boundary layer). Both types of solutions are expanded in a series with respect to the small parameter, and a recursion system of equations is constructed for terms of the series. The process of construction of approximations of the fundamental solution is called the first iteration process, the analogous process for rapidly changing solutions is called the second iteration process.

Let us construct these processes for determination of solution of Eq. (1, 1). Solutions of the second process just as solutions of the edge effect type must have a local character and must decrease rapidly with distance from the singular point. Such solutions will be referred to as local solutions.

Below, some properties of generalized homogeneous and adjoined homogeneous functions will be needed. We shall designate them by one letter P_i (the subscript i being equal to the degree of the function). The necessary information about these functions is presented in monograph [4].

A homogeneous function of the i th degree will be regular if all its derivatives of the order greater than i are equal to zero. Otherwise it will have a singularity at zero. Such a function will be called a singularity of the i th degree. All functions of the degree $i < 0$ will be singularities.

If some function has a singular point and in the vicinity of this point it is expanded in a series with respect to homogeneous (adjoined) functions, then the singularity of the lowest degree will be called the principal one.

The structure of the fundamental solution of the differential equation of the elliptic type is described in papers [5 and 6]. It is shown that the solution can be represented in the form of a sum of a singular and regular part, where the singular part is expanded in a series with respect to singularities the principal of which has the degree $2m - n$ ($2m$ is the degree of the equation).

2. As a preliminary, let us examine the equation with constant coefficients which is obtained by retaining in Eq. (1, 1) in operators M and L only terms with higher derivatives by equating coefficients to their values at the point $x = 0$, i. e. the singular point of the right side of the equation
$$\epsilon^s M_{2m,0}(\omega) + L_{2l,0}(\omega) = \delta \quad (2.1)$$

The solution of the equation with constant coefficients is simply determined with the aid of plane waves [4]
$$\xi = \omega_1 x_1 + \dots + \omega_n x_n$$

where ω_i are the components of the unit vector of the normal to the plane $\xi = \text{const}$. The Dirac function $\delta(x)$ is represented in the following manner [4]:

$$\delta(r) = c \int_{\omega} |\xi|^{-n} d\omega, \quad c = (-1)^{n/2} (2\pi)^{-n} (n-1)!$$

Here ω is the unit sphere, $d\omega$ is its element. This representation permits to find the solution of Eq. (2, 1) in the form

$$w = \int_{\omega} \Phi(\xi) d\omega \quad (2.2)$$

and to reduce the problem to the solution of the following ordinary differential equation:

$$\epsilon^s M_0 \Phi^{(2m)} + L_0 \Phi^{(2l)} = c |\xi|^{-n}$$

Here M_0 and L_0 are constants which depend on parameters ω_i and are determined by operators $M_{2m,0}$ and $L_{2l,0}$ in the transformation to the variable ξ . The solution of this equation consists of the sum solutions Φ_0 and Φ_1 of the following equations:

$$L_0 \Phi_0^{(2l)} = c |\xi|^{-n}, \quad L_0 \Phi_1^{(2m-2l)} + \epsilon^{-s} M_0^{-1} L_0^2 \Phi_1 = -c (|\xi|^{-n})^{(2m-4l)} \quad (2.3)$$

If in the right side $2m - 4l < 0$, the differentiation is replaced by integration. In this manner the initial equation separates into two. The first one does not contain the small parameter and corresponds to the degenerate equation $\epsilon = 0$, the second equation contains the small parameter. The solution of Eq. (2.3) can be constructed with the aid of the fundamental solution of the Ψ th equation and the latter is easily determined [4]. A fundamental solution of Eq. (2.3), decreasing for $|\xi| \rightarrow \infty$ such as is needed for the construction of the local solution, exists and is unique only when half of the roots λ in the characteristic equation $L_0 \lambda^{2(m-l)} + \epsilon^{-s} M_0^{-1} L_0^2 = 0$ has a positive real part and half negative. This condition is analogous to the condition of solutions of the edge effect type [1-3]. For equations of shells it is always satisfied. The solution for Ψ was constructed in paper [7]. The solution of equation Φ_1 is determined by convolution

$$\Phi_1 = -c \Psi * (|\xi|^{-n})^{(2m-4l)}$$

The solution of Eq. (2.1) will be

$$w = u' + w^* = \int_{\omega} \Phi_0 d\omega + \int_{\omega} \Phi_1 d\omega$$

The function w^* for $r = (x_1^2 + \dots + x_n^2)^{1/2} \rightarrow \infty$ can be presented in the form

$$u^* = \epsilon^s P_{\kappa} + O(\epsilon^{2s} P_{\kappa-1}), \quad \kappa = 2m - 4l - n \quad (2.4)$$

It grows if $\kappa > 0$ and decreases if $\kappa < 0$, but it has the coefficient ϵ^s and in some region $r > r_0$ (the quantity r_0 is discussed below) it is small in magnitude. Consequently, with some accuracy it can be neglected with respect to w' .

It is readily seen that the found solutions w' and w^* satisfy the equations

$$L_{2l,0}(u') = \delta, \quad \epsilon^s M_{2m,0} L_{2l,0}^{-1}(u^*) + u^* = -\epsilon^s L_{2l,0}^{-2} M_{2m,0}(\delta)$$

The operator L^{-k} is the inverse if L^k . It was shown with this example how it is formally possible to approach the splitting of the initial equation into two, and to separate the equation which determines the local solution. The solution of the second apparently complex equation can be constructed by the method of plane waves. In fact, the fundamental equation obtained by this method can be written in the following manner:

$$G = c \int_{\omega} \Psi' * |\xi|^{-n} d\omega, \quad \Psi' = \epsilon^s M_0 L_0^{-2} \Psi \quad (2.5)$$

With the aid of this function the local solution w^* is written differently

$$w^* = -\epsilon^s G * L_{2l,0}^{-2} M_{2m,0}(\delta) \quad (2.6)$$

In this manner the fundamental solution of Eq. (2.1) can be constructed in steps: at first it is determined for the degenerate equation, assuming $\epsilon = 0$. The principal singularity of this solution will be of the type P_{2l-n} , while the singularity of the initial equation

must be P_{2m-n} . This discrepancy is removed by adding the solution w^* , which according to (2.2) and (2.3) has the same principal singularity as the solution w' , but with the opposite sign. In fact, this can be seen if in the vicinity of the singular point the solution of Eq. (2.3) is obtained in the form of a power series in ξ and then integration is carried out over the sphere ω . The solution $w = w' + w^*$ will be the desired one with the required principal singularity.

The solution of Eq. (2.1) with constant coefficients can be represented in a different form which is more convenient for generalization to the case of variable coefficients

$$w = w' + w^* = \sum_{i=0}^k e^{i^s u_i'} + w_1^* \quad (2.7)$$

if the functions w_i' which satisfy the following recurrent system of equations:

$$L_{2l,0}(w_0') = \delta, \quad L_{2l,0}(w_{i+1}') = -M_{2m,0}(w_i'), \quad i = 0, 1, \dots, k-1$$

On the right side are the derivatives of generalized functions determined in [4]. From the condition that w be a solution of Eq. (2.1) the following equation is obtained for the function w_1^* :

$$e^s M_{2m,0}(w_1^*) + L_{2l,0}(w_1^*) = -e^{(k+1)s} M_{2m,0}(w_k')$$

Dividing the right and the left sides of this equation by the operator $L_{2l,0}$ according to the method described above, we arrive at the equation

$$e^s M_{2m,0} L_{2l,0}^{-1}(w_1^*) + w_1^* = -e^{(k+1)s} M_{2m,0} L_{2l,0}^{-1}(w_k') \quad (2.8)$$

The local solution of this equation according to (2.6) has the form

$$w_1^* = -e^{(k+1)s} G_s M_{2m,0} L_{2l,0}^{-1}(w_k')$$

The right side of Eq. (2.8) is a homogeneous function of degree $\gamma = 2l - n - 2k(m-l)$. The behavior of the function w_1^* for $r \rightarrow \infty$, as follows from the equation, is determined by this function, i. e.

$$w_1^* = e^{(k+1)s} P_\gamma + e^{(k+2)s} O(P_{\gamma-s})$$

If for $k = 0$ the function w_1^* increases when $r \rightarrow \infty$, then for $k \geq (2l - n) / s$ the function w_1^* will be decreasing and in this case the parameter ks enters into it with the power ks . Consequently, with increasing number of steps the local solution has more and more local character, i. e. it decays more intensively with distance from the singular point. It can be easily seen that the principal singularity of solution w will be the required one because, although the principal singularities of approximations w_i' increase with increasing number, the structure of solution w_1^* is such that "superfluous" singularities of function w' enter into w^* with the same coefficients but with opposite sign and are mutually eliminated in summation. This statement is verified by direct construction of functions w' and w^* .

In this manner the fundamental solution of Eq. (2.1) is represented in the form of a sum of functions w' and w^* which in analogy with [1-3] will be referred to as slowly varying and rapidly varying, respectively.

3. Now let us turn to the fundamental problem of determination of solution of Eq. (1.1).

The first process is formally constructed exactly in the same manner as in papers [1-3]

$$w' = w_0' + e^s w_1' + \dots + e^{ks} w_k' + R_1 \quad (3.1)$$

$$L(w_0') = \delta, \quad L(w_{i+1}') = -M(w_i'), \quad i = 0, 1, \dots, k-1$$

The series converges asymptotically everywhere with the exception of some region in the vicinity of the singular point in which it diverges because the order of the principal singularity of each subsequent approximation decreases.

By means of the second process we shall eliminate these discrepancies in the singularities of approximations of the first process and the divergence of the series in the vicinity of the singular point. Equations of the second process determine the rapidly varying part of the solution, the derivative of which is considerably greater than the function itself, i. e. $\partial w^* / \partial x_i \sim \varepsilon^{-1} w^*$. This relationship determines the form of equations of the second process.

In the vicinity of the singular point the variability of singularities may be great. In fact, on differentiation of such functions the degree of homogeneity decreases by one $\partial P_i / \partial x_j \sim r^{-1} P_i$. For $r \ll \varepsilon$ the inequality $\partial w^* / \partial x_i \gg \varepsilon^{-1} w^*$ is satisfied, i. e. instead of the first relationship used for the construction of the second process, a more stringent condition is satisfied. It is possible, however, to show that equations constructed for determination of solutions which satisfy the first condition include solutions which satisfy the more stringent second condition. Therefore, let us construct the second process formally in the same manner as it was done in papers [1-3], i. e. let us expand the coefficients of Eq. (1.1) in the vicinity of the singular point in a Taylor series

$$N(w) = \varepsilon^s \sum_{i=0}^{2m} \sum_{j=0}^{\infty} M_{ij}(w) + \sum_{i=0}^{2l} \sum_{j=0}^{\infty} L_{ij}(w) = \delta \quad (3.2)$$

Here the first index i indicates the order of the derivative in the operator, the second index j gives the degree of homogeneity of coefficients in this operator. Let us take k approximations of the first process and let us separate out the singular part W of this solution representing it in the form of a series with respect to singularities and let us write first terms of the series $W = \varepsilon^{ks} P_\gamma + \varepsilon^{k^2 s} P_{\gamma+1} + \dots + \varepsilon^{(k-1)s} P_{\gamma+2(m-l)} + \dots$

In the vicinity of the singular point the solution w^* in summation with W must annihilate the singularities which are written out. Let us represent the solution w in the form of the series

$$w^* = \sum_{i=0}^p w_i^* + R \quad (3.3)$$

considering that each subsequent approximation is ε^{-1} times greater than the previous one.

The sum of functions w^* and W must satisfy Eq. (3.2) near the singular point. Substituting this sum into (3.2) we obtain $N(w^*) = -N(W) + \delta$

We substitute (3.3) into (3.2) and equate terms of equal order with respect to magnitude. If in the process of equating it is taken into account that in the vicinity of the singular point the inequality $P_i \gg P_{i+1}$ holds (singularities of different degrees with respect to magnitude cannot be compared with each other and cannot appear in the same equation), we obtain the following recursion system of the second process for determination of approximations w_i^* :

$$\begin{aligned} \varepsilon^s M_{2m,0}(w_0^*) + L_{2l,0}(w_0^*) &= -\varepsilon^{(l+1)s} M_{2m,0}(P_\gamma) \\ \varepsilon^s M_{2m,0}(w_1^*) + L_{2l,0}(w_1^*) &= -\varepsilon^s M_{2m,1}(w_0^*) - L_{2l,1}(w_0^*) - \varepsilon^s M_{2m-1,0}(w_0^*) - \\ &- L_{2l-1,0}(w_0^*) - \varepsilon^{(k-1)s} [M_{2m,1}(P_\gamma) + M_{2m-1,0}(P_\gamma) + M_{2m,0}(P_{\gamma+1})], \dots \end{aligned} \quad (3.4)$$

On the left in all equations is the familiar (see 2) differential operator with constant

coefficients, on the right is the function which is known at each stage.

Elimination from functions of operator $L_{2l,0}$, which gives the slowly varying parts of solutions, is accomplished by method 2. The local solution of equations is determined for example by the convolution of fundamental solution (2.5) with the right sides. There are no principal difficulties in the solution.

Let us write the system of equations obtained for the first and second processes

$$L(w_0) = \delta, \quad L(w_{i+1}') = -M(w_i'), \quad i = 0, 1, 2, \dots, k-1$$

$$s^l L_{2l,0}^{-1} M_{2m,0}(w_j^*) + w_j^* = F_j, \quad j = 1, 2, \dots, p \quad (3.5)$$

Here F_j are the first parts of system (3.4) multiplied by operator $L_{2l,0}^{-1}$. The solution of Eq. (1.1) will be

$$w = \sum_{i=0}^k e^{is} w_i' + \sum_{i=0}^p w_i^* + R \quad (3.6)$$

It is now easy to see that "superfluous" singularities in solution w' are annihilated by solution w^* . This occurs in such a manner that the first approximation w_0^* eliminates the principal singularity at once in all approximations w_i' , the next approximation w_1^* eliminates the next singularity also in all approximations (this is verified by direct construction of principal singularities in solutions w' and w^* , for example by the method of plane waves). The number of iterations of the second approximation necessary for annihilation of "superfluous" singularities in the approximations of the first process is equal to $2(k+1)(m-l)+1$.

Asymptotic convergence of solution (3.6) with the exact solution is proven in a manner analogous to the procedure for the edge effect [3]. It is only necessary to clarify in what region near the singular point is it necessary to adopt the solution in the form (3.6). Outside this region it is necessary to neglect solutions of the second process.

It is evident from Eqs. (3.5) that with increasing distance from the singular point the behavior of local solutions undergoes a series of changes: the solutions transform from very rapidly varying and the $\partial w^* / \partial x_i \gg \varepsilon^{-1} w^*$ condition satisfying ones to rapidly varying ones (which are subject to the condition $\partial w^* / \partial x_i \sim \varepsilon^{-1} w^*$), and finally they change to slowly varying solutions for $r \rightarrow \infty$. Here the parameter ε enters to the $(k+1)s$ power, i. e. for $r \rightarrow \infty$ the approximations of the second process have a greater order of smallness than the k th approximation of the first process. In these discussions the concept of $r \rightarrow \infty$ must be understood conditionally and it must be assumed that $r > r_0$. The magnitude of radius r_0 can be given starting with the requirement that the function on this boundary of the singularity be already sufficiently slowly varying.

For example, if $\varepsilon = 1 \cdot 10^{-4}$, it is possible to take $r_0 = 100 \varepsilon$. Then in the vicinity of this boundary $\partial P_i / \partial x_j \sim 10^2 P_i \gg P_i$, consequently, inside the sphere of radius r_0 the singularities can be considered rapidly varying and therefore equations of the second process are true, and the solution represented by series (3.6) is applicable. On the boundary of the sphere the relationship $\varepsilon \partial P_i / \partial x_j \sim 10^{-2} P_i \ll P_i$ is satisfied, i. e. functions P_i are slowly varying and therefore outside the sphere the first process is convergent, and the solutions of the second process becomes unnecessary.

In this manner the radius r_0 is determined by the magnitude of the small parameter $r_0 = \zeta \varepsilon$. Here in the selection of coefficients ζ the following inequalities must be satisfied:

$$\zeta \gg 1, \quad (\zeta \varepsilon)^{-1} \gg 1$$

It is possible to find the number of iterations of the first and second processes inside

the sphere of radius r_0 which are necessary so that the solution (3.6) will approximate the exact solution with a preassigned asymptotic accuracy. The same thing can be done for the iteration of the first process outside the indicated sphere. We shall not deal with this here.

4. By the asymptotic method, the construction of the solution of a system of differential equations of the elliptic type (the degenerate system is of the same type) when the right side contains δ -functions or its derivatives, in principle does not differ from the case of one equation examined above.

In fact, in this paper in the search for the fundamental solution of one equation the first and second iterative processes were constructed such that in their form they were identical with iterations in the solution of boundary value problems [1-3]. In paper [1] the iterative processes were constructed for a system of equations in the theory of shells applicable to boundary value problems. From the development of arguments presented above for one equation it is possible to conclude that the process of construction of fundamental solutions of a system with the aid of the first and second processes is identical to the process developed in [1]. For this reason there is no need to repeat cumbersome derivations which will only repeat what was done before.

The first process is repeated rigorously [1]. Equations of the second process are constructed in the same manner as for one equation: the coefficients are expanded in a Taylor series, the solution is presented in the form of a series analogous to (3.6) and then together with the singular part of the solution of the first process it is substituted into the system. By equating values of the same order of smallness the system is transformed into a recursion system of systems of equations for determination of approximations of the second process, where at each stage one and the same system of equations with constant coefficients is solved. Here, of course, there are more difficulties than in the case of one equation; however they are purely technical.

We note that the approximations of the second process are determined independently of boundary conditions and for any equation or system they can be determined once and utilized for solution of various problems. It is only necessary that the singular point be located at a distance r_0 from the boundary.

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AN ALGORITHM OF THE SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Numerous problems of the theory of shells involve the solution of nonlinear boundary value problems [1 and 2] and this is often a fairly difficult task. Below we show, that in the number of cases numerical solutions of such problems are feasible.

Let us have the following system of differential equations with given boundary conditions:

$$dY_n(x)/dx = f_n(Y_n(x), q) \quad (1)$$

$$\text{where } \varphi_p(Y_n(0)) = 0 \text{ for } x = 0, \quad \psi_s(Y_n(1)) = 0 \text{ for } x = 1 \quad (2)$$

$$Y_n(x) = (y_1(x), \dots, y_n(x)), \quad f_n = (f_1, \dots, f_n)$$

$$\varphi_p(Y_n(0)) = (\varphi_1(Y_n(0)), \dots, \varphi_p(Y_n(0)))$$

$$\psi_s(Y_n(1)) = (\psi_1(Y_n(1)), \dots, \psi_s(Y_n(1))), \quad p + s = n \quad (3)$$

Here q is a parameter, and the type of solution depends on the numerical value of this parameter.

Let us replace some of the conditions given in (3) by conditions formulated in an integral form, e. g.

$$\int_0^1 F(Y_n(x)) dx + A = 0 \quad (4)$$

and let us introduce the following auxiliary function:

$$y_{n+1}(x) = \int_0^x F(Y_n(x)) dx + A \quad (5)$$

If the integrand function is continuous on $x \in [0, 1]$, we can write

$$dy_{n+1}/dx = F(Y_n(x)), \quad y_{n+1}(0) = A, \\ y_{n+1}(1) = 0 \quad (6)$$

Considering now the problem in an $(n + 1)$ -dimensional space, we arrive at the problem which was formulated above.

Solution of the problem (1)–(3) is obtained as follows. Keeping $q = q_0$

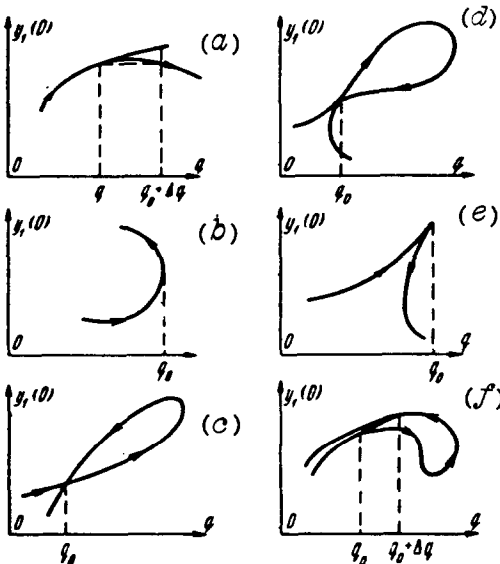


Fig. 1